Transition Maths and Algebra with Geometry

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Lecture Notes
Electrical and Computer Engineering

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Contents

Sequences

Sequence is an ordered collection of some elements. They are important because they appear everywhere. In particular, in computer science.

Definition

An infinite sequence is a function whose domain is the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. In some cases it will be useful to define sequences as functions with domain $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

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Notation and examples

Notation

Let $a : \mathbb{N} \to X$ be a sequence. Instead of writing $a(n)$ we will write a_n. We use the following equivalent notations to denote a sequence:

$$a_1, a_2, \ldots = \{a_n\}_{n=1}^{\infty} = \{a_n\}$$

Unless stated otherwise, we assume that $X = \mathbb{R}$, i.e. we deal with sequences of real numbers only.

Examples:

$$a_n = \sqrt{n},$$
$$b_n = 2^n.$$

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Arithmetic sequence

Definition

Let $a, d \in \mathbb{R}$ be two constants. A sequence $\{a_n\}$ of the form

$$\begin{aligned} a_1 &= a, \\ a_2 &= a + d, \\ &\dots \\ a_n &= a + (n-1)d \end{aligned}$$

is called an arithmetic sequence with the *initial value* a and *difference* d.

Fact

A sequence $\{a_n\}$ is an arithmetic sequence iff it is of the form

$$\begin{aligned} a_1 &= a, \\ a_n &= a_{n-1} + d \text{ for } n > 1 \end{aligned}$$

for some $a, d \in \mathbb{R}$.

Geometric sequence

Definition

Let $a, q \in \mathbb{R}$ be two constants. A sequence $\{a_n\}$ of the form

$$
\begin{aligned}
a_1 &= a, \\
a_2 &= a \cdot q, \\
&\ldots \\
a_n &= a \cdot q^{n-1}
\end{aligned}
$$

is called a geometric sequence with the *initial value* a and the *quotient* q.

Fact

A sequence $\{a_n\}$ is a geometric sequence iff it is of the form

$$
\begin{aligned}
a_1 &= a, \\
a_n &= a_{n-1} \cdot q \text{ for } n > 1
\end{aligned}
$$

for some $a, q \in \mathbb{R}$.

Monotonic sequences

Definition

A sequence $\{a_n\}$ is called *increasing* if

$$a_1 < a_2 < a_3 \ldots < a_{n-1} < a_n < \ldots$$

$\{a_n\}$ is *decreasing* if

$$a_1 > a_2 > a_3 \ldots > a_{n-1} > a_n > \ldots$$

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Monotonic sequences

Definition

A sequence $\{a_n\}$ is called *non-decreasing* if

$$a_1 \leq a_2 \leq a_3 \ldots \leq a_{n-1} \leq a_n \leq \ldots$$

$\{a_n\}$ is *non-increasing* if

$$a_1 \geq a_2 \geq a_3 \ldots \geq a_{n-1} \geq a_n \geq \ldots$$

Examples: $a_n = 0$ is non-increasing and non-decreasing. It is not increasing nor it is decreasing.
$b_n = n$ is increasing and non-decreasing. It is not decreasing nor it is non-increasing.

$c_n = (-1)^n$ is neither of the four.

Bounded sequences

Definition

A sequence $\{a_n\}$ is called *bounded* if there is a constant $M > 0$ such that

$$|a_n| < M \text{ for all values of n}$$

Equivalently,

$$-M < a_n < M \text{ for all values of n}$$

Example: $a_n = \frac{1}{n}$ is bounded. Indeed, put $M = 2$ and see that

$$-2 < \frac{1}{n} < 2 \text{ for any } n.$$

Now take $b_n = n$. It is not bounded.

Bounded sequences

Definition

A sequence $\{a_n\}$ is called *bounded from above* if there is a constant $M > 0$ such that

$$a_n < M \text{ for all values of n}$$

It is *bounded from below* if there is a constant $M > 0$ for which

$$-M < a_n \text{ for all values of n}$$

Fact

A sequence is bounded iff it is bounded from above and bounded from below.

Example: Take $b_n = n$. It is not bounded, but it is bounded from below.

Contents

Limits, informally

Informally, number L is the limit of a sequence $\{a_n\}$ if however close we get to L all but a finite number of terms of the sequence $\{a_n\}$ are even closer to L.

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Limits: definition

Definition

A number L is called *the limit* of a sequence $\{a_n\}$ if for any positive $\varepsilon > 0$ there is a natural number $N > 0$ such that for any $n > N$ the following inequality holds:

$$|a_n - L| < \varepsilon \text{ or equivalently } L - \varepsilon < a_n < L + \varepsilon$$

It is denoted by

$$\lim_{n \to \infty} a_n = L.$$

A sequence which has a limit is called *convergent*. Otherwise it is called *divergent*.

Limits: example

Consider the sequence $a_n = \frac{1}{n}$. Intuitively, we see that $\lim_{n\to\infty} \frac{1}{n} = 0$. Formally, fix $\varepsilon > 0$. We want to find $N > 0$ (depending on ε) such that

$$|a_n - L| < \varepsilon \text{ for all } n > N \equiv |\frac{1}{n} - 0| < \varepsilon$$

We solve the inequality $|\frac{1}{n}| < \varepsilon$:

$$|\frac{1}{n}| < \varepsilon \iff \frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$$

Hence, if we put N to be any natural number greater than $\frac{1}{\varepsilon}$ then for any $n > N$ we have

$$n > N > \frac{1}{\varepsilon}.$$

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Limits: properties

Definition

A sequence $\{b_n\}$ is said to be *a subsequence* of a sequence $\{a_n\}$ if it is obtained from $\{a_n\}$ by deleting some of the terms of $\{a_n\}$. In other words, if the terms of $\{b_n\}$ appear within the terms of $\{a_n\}$ in their given order.

Example: $\{b_n\} = 2, 4, 6, 8, \ldots$ is a subsequence of $\{a_n\} = 1, 2, 3, 4 \ldots$.

Theorem

A sequence $\{a_n\}$ converges to L iff all of its subsequences converge to L.

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Limits: properties

Theorem

A sequence $\{a_n\}$ converges to L iff all of its subsequences converge to L.

Example: Consider $a_n = (-1)^n$. $\{a_n\} = -1, 1, -1, 1, -1, \ldots$. We see that $b_n = 1$ and $c_n = -1$ are subsequences of a_n. Moreover,

$$\lim_{n\to\infty} b_n = 1,$$

$$\lim_{n\to\infty} c_n = -1.$$

Hence, a_n is NOT convergent.

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Limits: properties

Theorem

Assume that $\lim a_n = A$ and $\lim b_n = B$. Then

- $\lim(a_n + b_n) = A + B$,
- $\lim(a_n - b_n) = A - B$,
- $\lim(a_n \cdot b_n) = A \cdot B$,
- $\lim(c \cdot a_n) = c \cdot A$ for any constant $c \in \mathbb{R}$
- $\lim(\frac{a_n}{b_n}) = \frac{A}{B}$ if only $b_n \neq 0$ and $B \neq 0$.

Example $\lim \frac{1}{n^2} = \lim \frac{1}{n} \cdot \frac{1}{n} = 0 \cdot 0 = 0$.

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Sandwich Theorem

Fact

Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences. If $a_n \leq b_n$ for all but finite number of indices n then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

Sandwich Theorem

If $a_n \leq b_n \leq c_n$ for all but finite number of indices n and $\lim a_n = \lim c_n = L$ then

$$\lim_{n\to\infty} b_n = L.$$

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Sandwich Theorem

Consider

$$b_n = \frac{\cos(n)}{n^2}.$$

We know that $-1 \le \cos(n) \le 1$. Hence,

$$\frac{-1}{n^2} \le \frac{\cos(n)}{n^2} \le \frac{1}{n^2}$$

We know that

$$\lim_{n\to\infty} \frac{-1}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{\cos(n)}{n^2} = 0.$$

Boundedness monotonicity and convergence

Theorem

Every convergent sequence is bounded.

The converse is NOT true, but...

Theorem

Every monotonic and bounded sequence is convergent.

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Divergence to $\pm\infty$

Definition

We say that a sequence $\{a_n\}$ diverges to ∞ if for any $r > 0$ there is and index $N > 0$ such that

$$a_n > r \text{ for any } n > N$$

Similarly, we define divergence to $-\infty$. We denote it by $\lim_{n\to\infty} a_n = \infty$ (resp. $\lim_{n\to\infty} a_n = -\infty$).

Example: $a_n = -n^2$ is a sequence diverging to $-\infty$.

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Diverging sequences

Theorem

A non-decreasing sequence $\{a_n\}$ unbounded from above is divergent to ∞.

A dual statement is also true:

Theorem

A non-increasing sequence $\{a_n\}$ unbounded from below is divergent to $-\infty$.

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Diverging sequences: properties

Theorem

- if $\lim_{n\to\infty} a_n = \pm\infty$ then $\lim_{n\to\infty} \frac{1}{a_n} = 0$,
- if $\lim_{n\to\infty} a_n = \infty$ and $\{b_n\}$ is bounded from below then $\lim_{n\to\infty}(a_n + b_n) = \infty$,
- if $\lim_{n\to\infty} a_n = \infty$ and for any n a sequence $\{b_n\}$ satisfies $b_n \geq c$, where $c > 0$ is a constant then $\lim_{n\to\infty}(a_n \cdot b_n) = \infty$,
- if $\lim_{n\to\infty} a_n = \infty$ and for any n a sequence $\{b_n\}$ satisfies $a_n \leq b_n$ then $\lim_{n\to\infty} b_n = \infty$.

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Sequence $(1 + \frac{1}{n})^n$

Consider a sequence

$$a_n = (1 + \frac{1}{n})^n.$$

We calculate some its first few terms:

- $a_1 = 2$,
- $a_2 = \frac{9}{4} = 2.25$,
- $a_3 \approx 2.37$,
- $a_4 \approx 2.44$,
- $a_5 \approx 2.48$,
- $\ldots$,

Question

What does a_n converge to? Is it convergent at all?

Sequence $(1+\frac{1}{n})^n$

Recall that any monotonic and bounded sequence is convergent.

Fact

The sequence $a_n = \left(1+\frac{1}{n}\right)^n$ is monotonic and bounded. Hence, it is convergent.

Proof (monotonicity):
We will show that a_n is increasing ($a_{n+1} > a_n$ for any n). It is enough to show that the following inequality holds for any n:

$$\frac{a_n}{a_{n+1}} < 1.$$

Sequence $(1+\frac{1}{n})^n$

Proof (monotonicity): Indeed,

$$\frac{a_n}{a_{n+1}} = \frac{(1+\frac{1}{n})^n}{(1+\frac{1}{n+1})^{n+1}} = \frac{(1+\frac{1}{n})^n}{(1+\frac{1}{n+1})^n}\cdot\frac{1}{1+\frac{1}{n+1}} =$$

$$\left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}\right)^n\cdot\frac{n+1}{n+2} = \left(\frac{\frac{n+1}{n}}{\frac{n+2}{n+1}}\right)^n\cdot\frac{n+1}{n+2} = \left(\frac{(n+1)^2}{n\cdot(n+2)}\right)^n\cdot\frac{n+1}{n+2} =$$

$$\left(\frac{(n+1)^2}{(n+1)^2-1}\right)^n\cdot\frac{n+1}{n+2} = \left(\frac{(n+1)^2}{(n+1)^2-1}\right)^n\cdot\frac{n+1}{n+2} =$$

$$\left(\frac{1}{1-\frac{1}{(n+1)^2}}\right)^n\cdot\frac{n+1}{n+2}$$

Sequence $(1+\frac{1}{n})^n$

Proof (monotonicity):

$$\frac{a_n}{a_{n+1}} = \left(\frac{1}{1-\frac{1}{(n+1)^2}}\right)^n \cdot \frac{n+1}{n+2}.$$

Here, we apply the following well-known inequality

$$(1+x)^n \geq 1+nx.$$

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Sequence $(1+\frac{1}{n})^n$

We get:

$$\left(\frac{1}{1-\frac{1}{(n+1)^2}}\right)^n \cdot \frac{n+1}{n+2} \leq \left(\frac{1}{1-\frac{n}{(n+1)^2}}\right) \cdot \frac{n+1}{n+2} =$$

$$\frac{(n+1)^3}{(n+2)\cdot((n+1)^2-n)} = \frac{(n+1)^3}{(n+1)^3+1} < 1.$$

Sequence $(1 + \frac{1}{n})^n$

Hence,

$$\frac{a_n}{a_{n+1}} < 1.$$

This means that the sequence is increasing. It is also possible to prove that

$$0 < a_n < 3 \text{ for any } n.$$

In other words, it is possible to prove that $\{a_n\}$ is a bounded sequence.

$$\{a_n\} \text{ increasing and bounded } \implies \{a_n\} \text{ convergent}$$

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Number e

The limit of the sequence $\{a_n\}$ is denoted by e and is called the *Euler number*.

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Approximations of e may be calculated:

$$e = 2.718281828459045...$$

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Important limits

Limits

- if $a > 1$ then
$$\lim_{n\to\infty} a^n = \infty,$$
- if $|a| < 1$ then
$$\lim_{n\to\infty} a^n = 0,$$
- if $a > 0$ then
$$\lim_{n\to\infty} \sqrt[n]{a} = 1,$$
- $$\lim_{n\to\infty} \sqrt[n]{n} = 1,$$
- $$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Important limits: examples

Consider a sequence $(\frac{n-2}{n})^n$. We have

$$\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \to e^{-2}.$$

Now consider $\sqrt[n]{3n}$:

$$\sqrt[n]{3n} = \sqrt[n]{3}\sqrt[n]{n} \to 1 \cdot 1 = 1.$$

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Recursive definitions

Definition

A recursive definition of a sequence $\{a_n\}$ comprises:

- explicite definition of some first few terms,
- rule for calculating n-th from previous terms.

Example: $a_1 = 1$ and $a_n = n \cdot a_{n-1}$. The sequence is equal to

$$a_n = n! = 1 \cdot 2 \cdot 3 \ldots \cdot n.$$

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Recursive definitions

Example: $f_0 = 1$, $f_1 = 1$ and

$$f_n = f_{n-1} + f_{n-2}.$$

We obtain the following sequence:

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

This sequence is called *Fibonacci* sequence.

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Recursive definitions

Consider a sequence $\{x_n\}$ defined by

$$x_n = \sqrt{2x_{n-1} - 1}, \quad x_1 = 2.$$

Is it convergent? If so to what does it converge? Let us assume that $\{x_n\}$ is convergent (proof of this statement is left as a homework). Let $x_n \to L$. Hence, $x_{n-1} \to L$ and

$$L = \sqrt{2L - 1}$$

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Recursive definitions

The equation

$$L = \sqrt{2L - 1}$$

implies

$$L^2 = 2L - 1. \text{ Hence, } L^2 - 2L + 1 = 0.$$

The solution is $L = 1$.

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